JOURNAL OF APPROXIMATION THEORY 77, 65–73 (1994)

On the Definition and Normality of a General Table of Simultaneous Padé Approximants

BERNHARD BECKERMANN

Centre de Physique Théorique, CNRS, Luminy Case 907, F-13288 Marseille CEDEX 09, France and Laboratoire d'Analyse Numérique et d'Optimisation, UFR IEEA-M3, USTL, F-59655 Villeneuve d'Ascq CEDEX, France

JACEK GILEWICZ

Centre de Physique Théorique, CNRS, Luminy Case 907, F-13288 Marseille CEDEX 09, France and Université de Toulon, F-83130 La Garde, France

AND

VALERI KALIAGUINE

Centre de Physique Théorique, CNRS, Luminy Case 907, F-13288 Marseille CEDEX 09, France and Nizhnii Novgorod State University, Department of Mathematics, Gagarina 23, 606300 Nizhnii Novgorod, Russia

Communicated by D. S. Lubinsky

Received July 30, 1992; accepted in revised form October 27, 1992

We give a correct definition of a general table of simultaneous Padé approximants and study it's normality property.

1. DEFINITION OF A GENERAL TABLE OF SIMULTANEOUS PADÉ APPROXIMANTS

Throughout this paper we will assume that p is a positive integer, z_0 a complex number and that $f_1, ..., f_p$ are (formal) power series with respect to z_0

$$f_i(z) = \sum_{n=0}^{\infty} a_n^i (z - z_0)^n.$$

The tuple $(\mathbf{m}, \mathbf{n})$ will be called an index, if $\mathbf{m} = (m_1, m_2, ..., m_p)$, $\mathbf{n} = (n_1, n_2, ..., n_p)$, with m_i, $n_i + 1$ being non-negative integers ($i = 1, ..., p$).

0021-9045/94 $6.00

We define a denominator $Q(z)$ and numerators $P_1(z), P_2(z), ..., P_p(z)$ of simultaneous approximants associated with index $(\mathbf{m}, \mathbf{n})$ by the following relations: $m = m_1 + m_2 + \cdots + m_p$ $(i = 1, 2, ..., p)$

$$\deg Q \leqslant m, \qquad \deg P_i \leqslant n_i, \tag{1}$$

$$Q(z) f_i(z) - P_i(z) = c_i(z - z_0)^{m_i + n_i + 1} + \cdots, \tag{2}$$

Note that by definition it is sufficient to find a polynomial Q, $\deg Q \leqslant m$, with

$$(Qf_i)_j = 0, \qquad j = n_i + 1, n_i + 2, ..., n_i + m_i, \qquad i = 1, 2, ..., p \tag{3}$$

(we denote by $(g)_j$ the coefficient of $(z - z_0)^j$ in the power series of $g(z)$, especially $(g)_j = 0$ for $j < 0$). Then the polynomials $P_i(z)$ $(i = 1, 2, ..., p)$ are given by $P_i(z) = 0$ if $n_i = -1$, and otherwise by

$$P_i(z) = \sum_{k=0}^{n_i} (Qf_i)_k \, (z - z_0)^k. \tag{4}$$

A non-trivial polynomial $Q(z)$ exists for all indices $(\mathbf{m}, \mathbf{n})$ since (3) results in a system of m linear homogeneous equations with $(m+1)$ unknown coefficients of Q.

For the case $n_1 + m_1 = n_2 + m_2 = \cdots = n_p + m_p$, the index $(\mathbf{m}, \mathbf{n})$ will be called a *Mahler index*. Following the notations introduced in [11], the polynomials of simultaneous approximation Q, P_i corresponding to Mahler indices are so-called Hermite–Padé polynomials of type II. In [9], Hermite also defined a second class of Hermite–Padé polynomials of type I (confer, e.g., [7, 1, 8, 2, 12]). Close connections between both types have been pointed out in [11, 10, 4, 5]. In [6, 13], further results for type II polynomials are given, the case of diagonal Mahler indices $m_1 = m_2 = \cdots = m_p$, $n_1 = n_2 = \cdots = n_p$, is discussed in [3].

It is well known that in general neither the polynomials $Q(z)$ and $P_i(z)$ nor the vector of rational functions

$$\pi(\mathbf{m}, \mathbf{n}) = \left(\frac{P_1}{Q}, \frac{P_2}{Q}, ..., \frac{P_p}{Q}\right)$$

is unique for a fixed index $(\mathbf{m}, \mathbf{n})$. In the present paper we shall study the general table of simultaneous approximants defined as follows.

DEFINITION 1. For all indices $(\mathbf{m}, \mathbf{n})$ we shall call simultaneous Padé denominator a non-trivial polynomial $Q(z)$ of smallest degree verifying (1)–(2). The corresponding vector of rational functions $\pi(\mathbf{m}, \mathbf{n})$ will be called vector of simultaneous Padé approximants associated with index $(\mathbf{m}, \mathbf{n})$.

In this way the approximants $\pi(\mathbf{m}, \mathbf{n})$ are unique for all $(\mathbf{m}, \mathbf{n})$. To see it we suppose that it exist two polynomials $Q(z)$ and $Q^*(z)$ of degree as small as possible satisfying (1)–(2). Then the difference $Q(z) - aQ^*(z)$ verifies (1)–(2), too. By choosing a we can obtain $(Q - aQ^*) \leqslant \deg Q - 1$ and so Q is not of smallest degree verifying (1)–(2). Therefore $Q = aQ^*$ and $\pi(\mathbf{m}, \mathbf{n}) = \pi^*(\mathbf{m}, \mathbf{n})$. In this paper, the $2p$ dimensional table of the unique approximants $\pi(\mathbf{m}, \mathbf{n})$ will be called *simultaneous Padé table.*

As in the classical Padé case $(p = 1)$, we will show the quite important role of the approximants corresponding to normal indices in this table defined as follows.

DEFINITION 2. An index $(\mathbf{m}, \mathbf{n})$ is called a normal index if for all polynomials verifying (1)–(2) we have $\deg Q = m$, $\deg P_i = n_i$, and $Q(z_0) \neq 0$ (where the degree of the zero polynomial is defined to be -1).

If an index $(\mathbf{m}, \mathbf{n})$ is normal then obviously the vector $\pi(\mathbf{m}, \mathbf{n})$ is unique for all solutions of (1)–(2).

The conditions for a normal index $(\mathbf{m}, \mathbf{n})$ are all necessary for the fact that the approximant $\pi(\mathbf{m}, \mathbf{n})$ does not occur at any other position in the simultaneous Padé table. As a sufficient condition, we have to assume in addition that all coefficients c_i in (2) $(i = 1, 2, ..., p)$ are different from zero. This different type of regularity has been used in [6] to study Hermite–Padé polynomials of type II.

Let us consider for a moment a second system $\tilde{f}_1, ..., \tilde{f}_p$ of (formal) power series defined by $f_i(z) = (z - z_0)^{k_i} \cdot \tilde{f}_i(z)$ with $\tilde{f}_i(z_0) \neq 0$ $(i = 1, ..., p)$. It is easy to see that the simultaneous Padé table of both systems are closely connected, a fact which for $p > 1$ does not hold anymore if one only considers Mahler indices. Given an index $(\mathbf{m}, \mathbf{n})$ in the range $m_i + n_i < m + k_i$ for all $i = 1, ..., p$, the polynomials $Q(z) = (z - z_0)^m$, $P_i(z) = 0$, satisfy condition (1)–(2), hence the only normal index in this range is given by $\mathbf{m} = (0, ..., 0)$, $\mathbf{n} = (-1, ..., -1)$. Similar, if $(\mathbf{m}, \mathbf{n})$ is a normal index with $n_i < k_i$ for an i, then necessarily $m_i = 0$ and $n_i = -1$ which corresponds to the fact that we can consider the system of $p - 1$ (formal) power series $f_1, ..., f_{i-1}, f_{i+1}, ..., f_p$.

2. NORMAL INDICES

The usual way to study the approximants corresponding to normal indices is to introduce Hadamard determinants (see [7]). For $j, k, l \in N$

(the set of non-negative integers), $i=1, ..., p$, we denote by $A^i_k(j, l)$ the rectangular Toeplitz matrix of type $j \times k$

$$A^i_k(j,l)=\begin{pmatrix} a^i_l & a^i_{l-1} & \cdots & a^i_{l-k+1} \\ a^i_{l+1} & a^i_l & \cdots & a^i_{l-k+2} \\ \vdots & \ddots & \ddots & \vdots \\ a^i_{l+j-1} & a^i_{l+j-2} & \cdots & a^i_{l+j-k} \end{pmatrix}.$$

The system (3) of equations for the coefficients of the polynomial $Q(z) = u_0 + u_1(z-z_0) + \cdots + u_m(z-z_0)^m$ takes the following form:

$$\begin{pmatrix} A^1_{m+1}(m_1, n_1+1) \\ \vdots \\ A^p_{m+1}(m_p, n_p+1) \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5}$$

Introducing the determinants

$$H(\mathbf{m}, \mathbf{n}) = \begin{vmatrix} A^1_m(m_1, n_1) \\ \vdots \\ A^p_m(m_p, n_p) \end{vmatrix}, \qquad Q(\mathbf{m}, \mathbf{n}) = \begin{vmatrix} A^1_{m+1}(m_1, n_1+1) \\ \vdots \\ A^p_{m+1}(m_p, n_p+1) \\ 1\ (z-z_0) \cdots (z-z_0)^m \end{vmatrix} \tag{6}$$

we obtain the following lemmata describing normal indices.

LEMMA 1. *Problem* (3) *has a unique solution* (*up to multiplication with a constant*) *given by* $Q = const \cdot Q(\mathbf{m}, \mathbf{n})$ *if and only if* $Q(\mathbf{m}, \mathbf{n})$ *is non-trivial.*

Proof. The assertion follows immediately by application of the Kronecker–Capelli theorem on the system of linear Eqs. (5). ∎

Setting $\mathbf{e} = (1, 1, ..., 1)$, $\mathbf{e}_i = (0, 0, ..., 0, 1, 0, ..., 0)$ $(i = 1, 2, ..., p)$, we obtain for the denominator $Q(\mathbf{m}, \mathbf{n})$ and its corresponding numerators $P_i(\mathbf{m}, \mathbf{n})$

$$\begin{aligned} Q(\mathbf{m}, \mathbf{n})(z_0) &= (-1)^m \cdot H(\mathbf{m}, \mathbf{n}), \\ (Q(\mathbf{m}, \mathbf{n}))_m &= H(\mathbf{m}, \mathbf{n}+\mathbf{e}), \\ (P_i(\mathbf{m}, \mathbf{n}))_{n_i} &= (-1)^{m_i + m_{i+1} + \cdots + m_p} \\ &\quad \cdot H(\mathbf{m}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}-\mathbf{e}_i) \qquad (\text{if } n_i \geqslant 0), \\ (Q(\mathbf{m}, \mathbf{n}) \cdot f_i - P_i(\mathbf{m}, \mathbf{n}))_{m_i+n_i+1} &= (-1)^{m_{i+1} + \cdots + m_p} \\ &\quad \cdot H(\mathbf{m}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}). \end{aligned}$$

This yields the following determinantal characterization for normal indices.

LEMMA 2. *For $m > 0$, an index $(\mathbf{m}, \mathbf{n})$ is normal in the general simultaneous Padé table if and only if*

$$H(\mathbf{m}, \mathbf{n}) \neq 0, \quad H(\mathbf{m}, \mathbf{n}+\mathbf{e}) \neq 0, \quad \text{and} \quad H(\mathbf{m}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}-\mathbf{e}_i) \neq 0 \tag{7}$$

for all $i = 1, ..., p$ with $n_i \geq 0$. If $m = 0$, condition (7) has to be replaced by $a^i_{n_i} \neq 0$ for all $i = 1, ..., p$ with $n_i \geq 0$.

Note that for the case of classical Padé approximation ($p = 1$), condition (7) reads as follows

$$H(m, n) \neq 0, \qquad H(m, n+1) \neq 0, \qquad H(m+1, n) \neq 0, \tag{8}$$

which is the well-known characterization of the upper left corner of a square block in the corresponding c-table.

3. THE STRUCTURE OF THE SIMULTANEOUS PADÉ TABLE

One of our main results deals with the structure of the simultaneous Padé table. The block structure of classical Padé approximants ($p = 1$) is well known. In the general case we propose the following.

THEOREM 1. *Each approximant in the simultaneous Padé table coincides with an approximant corresponding to a normal index.*

The proof of this theorem is based on the following linear algebra lemmata.

LEMMA 3. *Let M be a matrix of order $(\sigma+1) \times \sigma$. By $M[i_1, ..., i_q]$ we denote the submatrix obtained from M by dropping the rows numbered $i_i, ..., i_q$. If* rank $M = \sigma$ *and if $i_1, ..., i_q$ are distinct numbers with* rank $M[i_\nu] < \sigma$, $\nu = 1, 2, ..., q$, *then* rank $M[i_1, ..., i_q] \leq \sigma - q$.

Proof (of Lemma 3). Let R_i denote the ith row of the matrix M. By assumption, there exists an i_0 with rank $M[i_0] = \sigma$. Moreover, we have the following non-trivial dependencies for the rows of M

$$\begin{cases} c_{1,1} R_1 + \cdots + c_{1,i_1-1} R_{i_1-1} + c_{1,i_1+1} R_{i_1+1} + \cdots + c_{1,\sigma+1} R_{\sigma+1} = 0 \\ c_{2,1} R_1 + \cdots + c_{2,i_2-1} R_{i_2-1} + c_{2,i_2+1} R_{i_2+1} + \cdots + c_{2,\sigma+1} R_{\sigma+1} = 0 \\ \vdots \quad \ddots \quad \vdots \qquad\qquad \vdots \qquad\qquad \ddots \quad \vdots \qquad\qquad \vdots \\ c_{q,1} R_1 + \cdots + c_{q,i_q-1} R_{i_q-1} + c_{q,i_q+1} R_{i_q+1} + \cdots + c_{q,\sigma+1} R_{\sigma+1} = 0 \end{cases}$$

In addition, let $c_{\nu,i_\nu} = 0$ for $\nu = 1, ..., q$. Note that the vectors $(c_{\nu,1}, ..., c_{\nu,\sigma})$ ($\nu = 2, ..., q$) must be multiplies of the vector $(c_{1,1}, ..., c_{1,\sigma})$, since otherwise by a linear combination we would be able to construct a relation $c_{0,1}R_1 + \cdots + c_{0,i_0-1}R_{i_0-1} + c_{0,i_0+1} + \cdots + c_{0,\sigma+1}R_{\sigma+1} = 0$ which contradicts the assumption rank $M[i_0] = \sigma$. Hence $c_{1,i_\nu} = 0$ for $\nu = 1, ..., q$ implying that the rows of the matrix $M[i_1, ..., i_q]$ are linearly dependent. ∎

For the index $(\mathbf{l}, \mathbf{k})$, $\mathbf{l} = (l_1, l_2, ..., l_p)$, $\mathbf{k} = (k_1, k_2, ..., k_p)$, and for $\sigma \in N$, let $A_\sigma(\mathbf{l}, \mathbf{k})$ denote the (rectangular) block Toeplitz matrix

$$A_\sigma(\mathbf{l}, \mathbf{k}) = \begin{pmatrix} A_\sigma^1(l_1, k_1) \\ \vdots \\ A_\sigma^p(l_p, k_p) \end{pmatrix}.$$

LEMMA 4. *Suppose that $(\mathbf{l}, \mathbf{k})$ is an index, $\sigma \in N$, with $\mathbf{l} = (l_1, l_2, ..., l_p)$ and $\sigma \leqslant l = l_1 + l_2 + \cdots + l_p$, such that*

$$\operatorname{rank} A_{\sigma+1}(\mathbf{l}, \mathbf{k} + \mathbf{e}) = \operatorname{rank} A_\sigma(\mathbf{l}, \mathbf{k} + \mathbf{e}) = \sigma.$$

Then we can find an $\mathbf{r} = (r_1, r_2, ..., r_p) \in N^p$, with $0 \leqslant r_i \leqslant l_i$ $(i = 1, ..., p)$ and $r_1 + r_2 + \cdots + r_p = \sigma$ such that rank $A_{\sigma+1}(\mathbf{r}, \mathbf{k} + \mathbf{e}) = \operatorname{rank} A_\sigma(\mathbf{r}, \mathbf{k} + \mathbf{e}) = \sigma$.

Proof (of Lemma 4). We show the assertion by recurrence on $l = l_1 + l_2 + \cdots + l_p$ for fixed σ.

If $l = \sigma$, then Lemma 4 is trivial, take $\mathbf{r} = \mathbf{l}$.

In the case $l > \sigma$, let without loss of generality $p' \leqslant p$ with $l_1 > 0, ..., l_{p'} > 0$, $l_{p'+1} = 0, ..., l_p = 0$ and let i'_ν denote the number of the last row of the νth (non-trivial) block of $A_{\sigma+1}(\mathbf{l}, \mathbf{k} + \mathbf{e})$ $(\nu = 1, ..., p')$. For notational conveniences, let

$$A = A_\sigma(\mathbf{l}, \mathbf{k} + \mathbf{e}), \qquad A' = A_\sigma(\mathbf{l}, \mathbf{k}).$$

Note that if there are rows numbered $j_1, ..., j_\tau$ $(\tau \geqslant 1)$ with rank $A[j_1, ..., j_\tau] = \sigma$ then also rank $A'[j_1, ..., j_\tau] = \sigma$ since otherwise we would have rank $A_{\sigma+1}(\mathbf{l}, \mathbf{k} + \mathbf{e}) > \sigma$.

Suppose that there exists a ν with rank $A[i'_\nu] = \sigma$. Then in view of

$$\begin{aligned} &A_{\sigma+1}(\mathbf{l}, \mathbf{k} + \mathbf{e})[i'_\nu] \\ &\quad = A_{\sigma+1}(\mathbf{l}', \mathbf{k} + \mathbf{e}) \qquad \text{with} \quad \mathbf{l}' = (l_1, ..., l_{\nu-1}, l_\nu - 1, l_{\nu+1}, ..., l_p), \end{aligned}$$

the assertion of Lemma 4 follows by recurrence.

Let us discuss the case where rank $A[i'_\nu] < \sigma$ for all $\nu = 1, ..., p'$. By assumption, there exist $j_1 > j_2 > \cdots > j_{l-\sigma}$ with rank $A[j_1, j_2, ..., j_{l-\sigma}] = \sigma$. We may assume that j_1 is minimal, i.e., for all $j < j_1$, $j \neq j_2, ..., j_{l-\sigma}$, we have rank $A[j, j_2, ..., j_{l-\sigma}] < \sigma$. Let i_ν be defined by

$$\{i_1, ..., i_q\} = \{1, 2, ..., j_1, i'_1, ..., i'_{p'}\} \setminus \{j_1, j_2, ..., j_{l-\sigma}\}$$

and hence $q \geqslant 1$. Consequently, rank $A[i_v, j_2, ..., j_{l-\sigma}] < \sigma$ for all $v = 1, ..., q$. From Lemma 3 we can conclude that $A[i_1, i_2, ..., i_q, j_2, ..., j_{l-\sigma}]$ is a matrix of size $(\sigma - q + 1, \sigma)$ having a rank less or equal to $\sigma - q$. This contradicts the fact that, because of the block Toeplitz structure, $A[i_1, i_2, ..., i_q, j_2, ..., j_{l-\sigma}]$ is a submatrix of the non-singular square matrix $A'[j_1, j_2, ..., j_{l-\sigma}]$.

Now we are able to continue the proof of Theorem 1.

Proof (of Theorem 1). Let $Q(z)$ be a denominator of simultaneous Padé approximants associated with index $(\mathbf{m}, \mathbf{n})$ normed such that

$$Q(z) = (z - z_0)^\gamma \cdot Q'(z) \qquad \text{with} \quad Q'(z_0) = 1 \tag{9}$$

with $\gamma \in N$ and let $P_1, ..., P_p$ denote the corresponding numerators. From (4) we can conclude that with Q also P_i is divisible by $(z - z_0)^\gamma$, let $P_i'(z) = P_i(z)/(z - z_0)^\gamma$. We define the indices $(\mathbf{m}', \mathbf{n}')$, $(\mathbf{l}, \mathbf{k})$ by $(i = 1, ..., p)$

$$\mathbf{n}' = (n_1', ..., n_p') \qquad \text{with} \qquad n_i' = \deg P_i \geqslant -1, \tag{10}$$

$$\mathbf{m}' = (m_1', ..., m_p') \qquad \text{with} \qquad m_i' = \begin{cases} \max\{\gamma - n_i - 1, m_i\} \geqslant 0 & \text{if} \quad P_i = 0, \\ m_i \geqslant 0 & \text{if} \quad P_i \neq 0, \end{cases} \tag{11}$$

$$\mathbf{l} = (l_1, ..., l_p) \qquad \text{with} \qquad l_i = \begin{cases} m_i' + n_i - n_i' - \gamma \geqslant 0 & \text{if} \quad P_i = 0, \\ m_i' + n_i - n_i' \geqslant 0 & \text{if} \quad P_i \neq 0, \end{cases} \tag{12}$$

$$\mathbf{k} = (k_1, ..., k_p) \qquad \text{with} \qquad k_i = \begin{cases} n_i' = -1 & \text{if} \quad P_i = 0, \\ n_i' - \gamma = \deg P_i' \geqslant 0 & \text{if} \quad P_i \neq 0, \end{cases} \tag{13}$$

such that $\mathbf{l} + \mathbf{k} = \mathbf{m}' + \mathbf{n} - (\gamma, ..., \gamma)$. Finally, let $d := \deg Q$. By definition, Q is also a simultaneous Padé denominator associated with the indices $(\mathbf{m}', \mathbf{n})$ and $(\mathbf{m}' + \mathbf{n} - \mathbf{n}', \mathbf{n}')$. Moreover, $Q, P_1, ..., P_p$ is (up to multiplication with a constant) the unique solution of (1)–(2) with index $(\mathbf{m}' + \mathbf{n} - \mathbf{n}', \mathbf{n}')$ where the degree of the denominator is less or equal to d. Setting $Q(z) = u_0 + u_1 \cdot (z - z_0) + \cdots + u_d \cdot (z - z_0)^d$ $(u_0 = u_1 = \cdots = u_{\gamma-1} = 0)$, we see that the two systems of linear equations

$$A_{d+1}(\mathbf{m}' + \mathbf{n}' - \mathbf{n}, \mathbf{n}' + \mathbf{e}) \cdot (x_0, ..., x_d)^T = (0, ..., 0)^T \tag{14}$$

$$A_{d-\gamma+1}(\mathbf{l}, \mathbf{k} + \mathbf{e}) \cdot (x_\gamma', ..., x_d')^T = (0, ..., 0)^T \tag{15}$$

have (up to multiplication with a constant) the unique solutions $(u_0, ..., u_d)$, and $(u_\gamma, ..., u_d)$, respectively. Since $u_d \neq 0$, we may apply Lemma 4 in order to obtain a vector $\mathbf{r} = (r_1, ..., r_p)$, $0 \leqslant r_i \leqslant l_i$, $i = 1, ..., p$, $r_1 + \cdots + r_p = d - \gamma$, such that the determinant of the square submatrix $A_{d-\gamma}(\mathbf{r}, \mathbf{k} + \mathbf{e})$ of $A_{d-\gamma+1}(\mathbf{l}, \mathbf{k} + \mathbf{e})$ does not vanish. Lemma 1 implies that

Q', $P'_1, ..., P'_p$ is (up to multiplication with a constant) the unique solution of (1)–(2) with index $(\mathbf{r}, \mathbf{k})$. By construction we have $Q'(z_0) \neq 0$, $\deg Q' = d - \gamma$ and $\deg P'_i = k_i$, $i = 1, ..., p$. Hence $(\mathbf{r}, \mathbf{k})$ is a normal index with

$$\pi(\mathbf{r}, \mathbf{k}) = \pi(\mathbf{m}, \mathbf{n}) = \left(\frac{P'_1}{Q'}, ..., \frac{P'_p}{Q'}\right)$$

which yields the assertion of Theorem 1. ∎

As a consequence of Theorem 1, each Hermite–Padé approximant of type II with minimal degree of denominator has a determinant representation. A similar result has been obtained in [1, p. 195] for type I approximants using a more general definition of minimal degree.

EXAMPLE. Let as in [3, Example 2]

$$f_1(z) = 2z^6 + \frac{1+z}{1-z}, \qquad f_2(z) = 2z^6 + \frac{1+z}{1-2z}, \qquad z_0 = 0,$$

then we observe the following facts:

(a) A corresponding normal index is not unique, for instance for the approximant

$$\pi((4,4),(4,4)) = \left(\frac{z^2(3-4z-4z^2)}{z^2(3-10z+10z^2-6z^3+6z^4-6z^5)}, \frac{z^2(3-z-2z^2)}{z^2(3-10z+10z^2-6z^3+6z^4-6z^5)}\right),$$

we find the normal indices ((4, 1), (2, 2)), and ((1, 4), (2, 2)).

(b) Note that the index of the approximant of (a) is a Mahler index, i.e., $n_1 + m_1 = n_2 + m_2 = \cdots = n_p + m_p$. In contrast, this condition does not hold for any of the corresponding normal indices.

(c) In general we will not have a block structure in form of generalized squares or rectangulars, since for example

$$\begin{aligned} \pi((4,4),(4,4)) &= \pi((4,1),(2,2)) = \pi((1,4),(2,2)) \neq \pi((2,2),(2,2)) \\ &= \pi((3,2),(2,2)) = \pi((2,3),(2,2)) = \pi((2,2),(4,4)) \\ &= \left(\frac{(1+z)(1-2z)}{(1-z)(1-2z)}, \frac{(1+z)(1-z)}{(1-z)(1-2z)}\right). \end{aligned}$$

Remark. We do not know exactly a geometric form of "blocks" in our $2p$ dimensional simultaneous Padé table. Some simple examples show that their shape might be quite complicated, it depends on specific properties of the functions $f_1(z), f_2(z), \ldots, f_p(z)$ (for the special case of diagonal Mahler indices, a block structure was investigated in [3]). In contrast, the structure of a table of Hermite–Padé type I approximants was given in [2].

REFERENCES

1. B. BECKERMANN, "Zur Interpolation mit polynomialen Linearkombinationen beliebiger Funktionen," Thesis, Univ. Hannover, 1990.
2. B. BECKERMANN, The structure of the singular solution table of the M-Padé approximation problem, *J. Comput. Appl. Math.* **32**, Nos. 1–2 (1990), 3–15.
3. B. BECKERMANN AND G. LABAHN, A uniform approach for Hermite Padé and simultaneous Padé Approximants and their Matrix-type generalizations, *Numerical Algorithms* **3** (1992), 45–54.
4. J. COATES, On the algebraic approximation of functions, I–III, *Indag. Math.* **28** (1966), 421–461.
5. J. COATES, On the algebraic approximation of functions, IV, *Indag. Math.* **29** (1967), 205–215.
6. M. G. DE BRUIN, Some aspects of simultaneous rational approximation, *in* "Numerical Analysis and Mathematical Modelling," Banach Center Publications, Vol. 24 (PWN-Polish Scientific Publishers, Warsaw, 1990), pp. 51–84.
7. J. D. DORA AND C. D. CRESCENDO, Approximants de Padé–Hermite, *Numer. Math.* **43** (1984), 23–39.
8. A. DRAUX AND A. MAANAOUI, Vector orthogonal polynomials, *J. Comput. Appl. Math.* **32** (1990), 59–68.
9. C. HERMITE, Sur la Généralisation des Fractions Continues algébriques, *in* "Oeuvres," Vol. 4, pp. 357–377, Gauthier-Villars, Paris, 1917.
10. H. JAGER, A multidimensional generalization of the Padé table, *Indag. Math.* **26** (1964), 193–249.
11. K. MAHLER, Perfect systems, *Compositio Math.* **19** (1968), 95–166.
12. M. VAN BAREL AND A. BULTHEEL, The computation of non-perfect Padé–Hermite approximants, *Numerical Algorithms* **1** (1991), 285–304.
13. J. VAN ISEGHEM, Vector Padé approximants, *in* "Numerical Mathematics and Applications" (R. Vichnevetsky and J. Vignes, Eds.), pp. 73–77, North-Holland, Amsterdam, 1986.